THE METAPLECTIC CASSELMAN-SHALIKA FORMULA

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Abstract. This paper studies spherical Whittaker functions for central extensions of reductive groups over local fields. We follow the development of Chinta-Offen to produce a metaplectic Casselman-Shalika formula for tame covers of all unramified groups.

1. Introduction

Suppose that G is an unramified reductive group over a non-archimedean local field F. This means that G is the generic fibre of a smooth reductive group scheme over the ring of integers O_F, or equivalently that G is quasi-split and splits over an unramified extension of F. The Casselman-Shalika formula is an explicit formula for the Whittaker function that is associated to the unramified principal series of $G(F)$.

In this paper, we replace G be a central extension by a finite cyclic group, and develop a Casselman-Shalika formula for this so-called metaplectic group. Our main result is the union of Theorem 8.1, Proposition 13.1 and Proposition 14.1. The latter propositions detail how to compute the Weyl group action appearing in the metaplectic Casselman-Shalika formula that is Theorem 8.1.

Our approach is to follow the technique of Chinta and Offen [CO] who have shown how to generalise the approach of Casselman and Shalika [CS] to provide a formula for the Whittaker function on the metaplectic cover of GL_r. The purpose of this paper is to show how their technique generalises to cover the more general case of covers of unramified groups.

This paper can be considered to consist of two parts. In the first part of this paper, we work in the generality of considering any finite cyclic cover of any reductive G. This culminates in the aforementioned Theorem 8.1, and closely follows the approach of Chinta and Offen. The second part begins with Section 9 and is new, developing the necessary extra results to enable one to compute this Whittaker function in the case where the underlying reductive group is unramified.

To conclude, we compare our computation of the metaplectic Whittaker function with the objects appearing in the local part of a Weyl group multiple Dirichlet series constructed by Chinta and Gunnells [CG].

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2. the metaplectic group

Fix a positive integer n. Let F be a non-archimedean local field with valuation ring O_F, uniformiser ϖ and residue field of order q which we assume to be congruent to 1 modulo $2n$. Let G be a connected reductive algebraic group over F. Let S be

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a maximal split torus of G and let T be a maximal torus of G containing S. We use the following theorem to find a split reductive subgroup G' of G that will be of use to us.

Theorem 2.1. [BT, Théorème 7.2] *Let $\Phi = \Phi(S, G)$ be the root system of G relative to S, and let Φ' be the set of roots a for which $2a$ is not a root. Let Δ be a choice of simple roots in Φ'. For each $a \in \Delta$, let E_a be a one-dimensional subgroup of the root subgroup of a, and let V be the group generated by the E_a. Then G possesses a unique split reductive subgroup G' containing S and V. The torus S is a maximal torus of G' and the root system is $\Phi' = \Phi(G', S)$. In particular, the Weyl groups of G and G' are isomorphic.*

We write W for the Weyl group of G and $W_{\overline{F}}$ for the Weyl group of $G_{\overline{F}}$. Consider the geometric cocharacter lattice $X_*(T_{\overline{F}}) = \operatorname{Hom}_{\overline{F}}(\mathbb{G}_m, T_{\overline{F}})$. This is equipped with actions of both $W_{\overline{F}}$ and the Galois group $\operatorname{Gal}(\overline{F}/F)$. Let Q be a $\operatorname{Gal}(\overline{F}/F)$ and $W_{\overline{F}}$-invariant integer valued quadratic form on $\operatorname{Hom}_{\overline{F}}(\mathbb{G}_m, T_{\overline{F}})$.

By the work of Brylinski and Deligne [BD], associated to Q is a central extension of G by K_2 as sheaves on the big Zariski site over $\operatorname{Spec}(F)$. At the level of F-points, this gives a central extension of G by $K_2(F)$. We push forward this central extension by the Hilbert symbol $K_2(F) \to \mu_n$ to obtain a central extension $\widetilde{G}$ of G by μ_n. Explicitly, there is a short exact sequence of topological groups

$$1 \to \mu_n \to \widetilde{G} \to G \to 1$$

with μ_n lying in the centre of $\widetilde{G}$. On occasion, we shall find it necessary to express $\widetilde{G}$ in terms of a 2-cocycle on G. When this is the case, we typically denote the section $G \mapsto \widetilde{G}$ by $\mathbf{s}$ and the 2-cocycle by σ. Note that $\mathbf{s}$ is not a homomorphism, the multiplication in $\widetilde{G}$ is given by $\mathbf{s}(g_1g_2) = \mathbf{s}(g_1)\mathbf{s}(g_2)\sigma(g_1, g_2)$.

For any subgroup H of G, we denote by $\widetilde{H}$ the inverse image of H in $\widetilde{G}$.

Let $B(x, y) = Q(x + y) - Q(x) - Q(y)$ be the bilinear form on $X_*(T_{\overline{F}})$ associated to Q. The commutator map $[\cdot, \cdot] : \widetilde{G} \times \widetilde{G} \longrightarrow \mu_n$ gives a well-defined map from $G \times G$ to μ_n. When restricted to T, it takes the following form [BD, Corollary 3.14]

$$[x^\lambda, y^\mu] = (x, y)^{B(\lambda,\mu)}.$$

Let us now restrict our attention to the central extension $\widetilde{G}'$ of G'. There is a natural inclusion of cocharacter groups $X_*(S) \subset X_*(T_{\overline{F}})$. The restriction of Q to $X_*(S)$ is a W-invariant quadratic form and $\widetilde{G}'$ is the central extension of G' associated to Q by the Brylinski-Deligne theory.

Since G' is split, we can, and will make use of the theory developed in [Mc2] where only covers of split groups were considered.

We may identify the Bruhat-Tits building of G' with a subset of the Bruhat-Tits building of G. Pick a hyperspecial point in the apartment corresponding to S, and let $\mathbf{G}$ be the corresponding group scheme over O_F with special fibre G via Bruhat-Tits theory. We let $K = \mathbf{G}(O_F)$, this is a maximal compact subgroup of G.

We will need to define a lift for any element of W into G. To achieve this, it suffices to work inside the group G'. By a theorem of Tits [T], for each simple reflection $s_\alpha \in W$, we can choose $w_\alpha \in \mathbf{G}'(O_K)$ such that the collection of w_α's so obtained satisfy the braid relations. Furthermore, if we consider the natural projection from the group generated by the w_α to W, the kernel is an elementary abelian 2-group contained in S. Thus for any $w \in W$, we define a lift by writing $w = s_{\alpha_1} \cdots s_{\alpha_N}$ as a

reduced product of simple reflections, and letting the lift be the product $w_{\alpha_1} \cdots w_{\alpha_N}$. Let us denote by W_0 the subgroup of $\widetilde{G}$ generated by the w_α.

At all stages, unless explicitly mentioned otherwise, we choose normalisations of Haar measures such that the intersection with the maximal compact subgroup K has volume 1.

We now discuss a couple of splitting properties. A subgroup J of G is said to be split in the extension if there is a section $J \mapsto \widetilde{J}$ that is a group homomorphism. When this occurs, we also denote the image of J in $\widetilde{G}$ by J.

Theorem 2.2. [Mc2, Proposition 4.1] *Any unipotent subgroup of G has a canonical splitting.*

We will work under the following assumption.

Assumption 2.3. *The subgroup K has a splitting.*

The author does not know of any situation subject to the already imposed condition of $2n$ dividing $q-1$ where this assumption are not satisfied. We will now explain why making this assumption does not involve any loss of generality when G is unramified.

Firstly, note that this splitting property is known to be true in the split case [Mc2, Theorem 4.2]. Now let us note that it is possible to pushforward the central extension $\widetilde{G}$ by the inclusion $\mu_n \mapsto S^1$ and get a central extension of G by the group of complex numbers of norm 1. Doing so does not in any way change the representation theory of $\widetilde{G}$. Working with the extension by S^1, we can immediately descend the splitting in the split case to the unramified case.

The splitting of K is not unique in general. We will choose once and for all such a splitting. Note that this allows us to define a lift of any element $w \in W$ to an element of $\widetilde{G}$, which we will also by abuse of notation call w. We also choose a lift of $X_*(S)$ into $\widetilde{G}$ (which exists as $2n$ divides $q-1$), denoted by $\lambda \mapsto \varpi^\lambda$.

Let M be the centraliser of S in G. This is a minimal Levi subgroup of G. Now that we have a splitting of K, we define a subgroup H as follows

$$H = \{h \in \widetilde{M} \mid [h, \eta] \subset K \;\forall\, \eta \in \widetilde{M} \cap K\}.$$

If G happens to be quasisplit, then this is simply the centraliser in $\widetilde{M}$ of $M \cap K$.

For each coroot α, we define the integer $n_\alpha = n/\gcd(n, Q(\alpha))$. This has the consequence that $\varpi^{n_\alpha \alpha} \in H$.

Write Λ' for the lattice $\widetilde{M}/(\mu_n \times (M \cap K))$ and Λ for the finite index sublattice $H/(\mu_n \times (H \cap K))$. If G is unramified, then Λ' is canonically isomorphic to the cocharacter lattice $X_*(S)$.

We will make one more assumption, which again is unnecessary in the unramified case.

Assumption 2.4. *The quotient group $H/(M \cap K)$ is abelian.*

This has the consequence that there is an isomorphism of groups $H/(M \cap K) \cong \mu_n \times \Lambda$.

In the unramified case, the following lemma proves that this assumption is always satisfied.

Lemma 2.5. *If G is unramified, then H is abelian.*

Proof. Since G is unramified, we have M is abelian and $M = S(M \cap K)$. Now suppose that $h_1, h_2 \in H$. Write $h_i = s_i k_i$ with $s_i \in \widetilde{S}$ and $k_i \in M \cap K$. The only nontrivial part is to show that s_1 and s_2 commute. But s_i commutes with $\widetilde{S} \cap K$, so by [Mc2, Lemma 5.3], we're done. □

One may wish to compare these conditions we have shown to hold in the unramified case with those appearing in [L, Définition 3.1.1].

3. UNRAMIFIED REPRESENTATIONS

Let P be a minimal parabolic F-subgroup of G containing S and let U be its unipotent radical. The quotient P/U is canonically isomorphic to the Levi subgroup $M = Z_G(S)$.

Let χ be a complex-valued point of $\mathrm{Spec}\,(\mathbb{C}[\Lambda])$, or equivalently a $\mathbb{C}$-valued character of Λ. We define the corresponding unramified representation $(\pi_\chi, i(\chi))$ of $\widetilde{M}$ as follows. Given χ, we turn it into a character of $\mu_n \times \Lambda$ by letting μ_n act faithfully. In this way, χ defines a one-dimensional representation of the subgroup H as this canonically surjects onto $\mu_n \times \Lambda$. We define $i(\chi)$ to be the induction of this representation from H to $\widetilde{M}$. Note that $i(\chi)$ is finite dimensional.

We also construct an unramified principal series representation $I(\chi)$ of $\widetilde{G}$. First we use the canonical surjection $\widetilde{P} \to \widetilde{M}$ to consider $i(\chi)$ as a representation of $\widetilde{P}$. The unramified principal series representation $I(\chi)$ is now defined to be the induction of this representation from $\widetilde{P}$ to $\widetilde{G}$.

Concretely $I(\chi)$ is the space of all locally constant functions $f : \widetilde{G} \longrightarrow i(\chi)$ such that

$$f(pg) = \delta^{1/2}(p)\pi_\chi(p)f(g)$$

for all $p \in \widetilde{P}$ and $g \in \widetilde{G}$ where δ is the modular quasicharacter of $\widetilde{P}$. The action of $\widetilde{G}$ on $I(V)$ is given by right translation.

Lemma 3.1. *The kernel of the projection from W_0 to W lies in the centre of $\widetilde{M}$.*

Proof. Suppose z is in the aforementioned kernel. Since $p(z) \in S$ which is central in M, there is a group homomorphism from $\widetilde{M}$ to μ_n given by $m \mapsto [m, z]$. Since $\widetilde{M}$ is the union of the conjugates of $\widetilde{S}$, it suffices to show that this homomorphism is trivial on $\widetilde{S}$. But to show this, it suffices to pass to a field extension where $\widetilde{G}$ is split. Using our formulae for the commutator in the split case, we see that z is central in $\widetilde{S}$ as we are assuming that $(q-1)/n$ is even, and we are done. □

Now W_0 acts on $\widetilde{M}$ by conjugation, and the above lemma shows that this descends to an action of W on $\widetilde{M}$. This induces an action of W on the category of representations of $\widetilde{M}$. In particular, we have constructed an explicit isomorphism between the underlying vector spaces of $i(\chi)$ and $i(w\chi)$ for any χ and any $w \in W$.

Theorem 3.2. *The map $f \mapsto f(1)$ is an isomorphism between $I(\chi)^K$ and $i(\chi)^{\widetilde{M} \cap K}$. These are both one-dimensional vector spaces.*

Proof. The argument of [Mc2, Lemma 6.3] applies in this case without change. □

The identification of the spaces $i(\chi)$ and $i(w\chi)$ constructed above can be construed as an action of W on $i(\chi)$. Under this action, the subspace $i(\chi)^{\widetilde{M} \cap K}$ is invariant.

Let us pick a non-zero vector v_0 in this subspace. By Theorem 3.2, we choose a spherical vector $\phi_K^{(\chi)} \in I(\chi)^K$ for each χ in a W-orbit in a compatible manner such that $\phi_K^{(w\chi)}(1) = v_0$.

4. Intertwining operators

For any $w \in W$, define U_w to be the quotient $U/(U \cap wUw^{-1})$. The (unnormalised) intertwining operators $T_w : I(\chi) \longrightarrow I(\chi^w)$ are defined by

$$(T_w f)(g) = \int_{U_w} f(w^{-1}ug)du.$$

whenever this is absolutely convergent, and by a standard process of meromorphic continuation in general, for example following [Mc2, §7]. It is a routine calculation to show that T_w does indeed map $I(\chi)$ into $I(w\chi)$ as claimed. We denote by X the open subset of $\mathrm{Spec}\,(\mathbb{C}[\Lambda])$ on which all the intertwining operators T_w have no poles.

When $w = s$ is a simple reflection, then we freely identify U_s with the intersection of U and the corresponding standard Levi subgroup M_s.

Proposition 4.1. *Suppose that w_1 and w_2 are two elements of W such that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. Then the intertwining operators satisfy the identity $T_{w_1w_2} = T_{w_1}T_{w_2}$.*

Proof. There is a measure preserving bijection from $U_{w_1} \times U_{w_2}$ to $U_{w_1w_2}$ given by $(u_1, u_2) \mapsto w_1u_2w_1^{-1}u_1$. The remainder of the proof is a standard manipulation involving Fubini's theorem. □

Let B denote the fraction field of the ring $\mathbb{C}[\Lambda]$.

Theorem 4.2. *There exists a non-zero element $c_w(\chi) \in B$ such that for all characters χ of $\mathbb{C}[\Lambda]$,*

$$T_w\phi_K^{(\chi)} = c_w(\chi)\phi_K^{(w\chi)}.$$

Remark 4.3. In Theorem 12.1, we will provide a more precise statement for unramified groups.

Proof. By Proposition 4.1, we may assume without loss of generality that w is a simple reflection s. Also by meromorphic continuation, we may assume without loss of generality that the defining integral for T_s converges.

The function $T_s\phi_K$ is guaranteed to be K-invariant, and hence by Theorem 3.2 is a scalar multiple of $\phi_K^{(s\chi)}$. Thus $c_s(\chi)$ exists as a function on X. To evaluate it, it suffices to evaluate $(T_w\phi_K)(1)$.

There is a filtration on U_s induced by a valuation of root datum. This consists of the data of a compact subgroup $U_{s,r}$ of U_s for each $r \in \mathbb{R}$ with the property that $U_{s,r'} \subset U_{s,r}$ if $r' \leq r$. We let $C_r = U_{s,r} \setminus \cup_{r'<r} U_{s,r}$. First we collate some facts about these sets.

If $u \in C_r$ and $r > 0$, then $s^{-1}u \in \mu_n\varpi^{r\alpha}U_sK$. Conjugation by ϖ^α sends C_r to C_{r+2}, and there are only finitely many orbits of non-empty C_r under the action of conjugation by ϖ^α.

There is a decomposition of U_s into the disjoint union

$$U_s = (U_s \cap K) \bigcup \left(\bigcup_{r>0} C_r \right).$$

We apply this decomposition to the domain of integration in the equation

$$T_s\phi_K(1) = \int_{U_s} \phi_K(s^{-1}u)du,$$

The integral over $U_s \cap K$ is equal to v_0. For the integrals over the C_r, we use the fact that $T_s\phi_K^{(\chi)}(1) \in i(\chi)^{\widetilde{M}\cap K}$ to see that only those r for which $\varpi^{r\alpha} \in H$ can give a non-zero contribution. This provides an expression for $c_s(\chi)$ as an infinite sum of elements of B.

In passing from the integral over C_r to that over C_{r+2n_α} via conjugation by ϖ^{n_α}, the integrand has been multiplied by $(\delta^{1/2}\chi)(\varpi^{2n_\alpha\alpha})$ while the change of coordinates contributes a factor of $\delta(\varpi^{n_\alpha\alpha})^{-1}$. Hence one integral is $\chi(\varpi^{2n_\alpha})$ times the other. Thus our expression for $c_s(\chi)$ is actually a geometric series, so is an element of B as required.

One simple way to see that $c_s(\chi)$ is non-zero is to take the limit as x_α tends to zero, when only the integral over $U_s \cap K$ survives. $\square$

In light of the above result, we now define a renormalised version of the intertwining operators. Let $\overline{T}_w = c_w(\chi)^{-1}T_w$. The upshot is that we have the equation

$$\overline{T}_w\phi_K^{(\chi)} = \phi_K^{(w\chi)} \tag{4.1}$$

as well as the following Proposition.

Proposition 4.4. *For any $w_1, w_2 \in W$, we have*

$$\overline{T}_{w_1w_2} = \overline{T}_{w_1}\overline{T}_{w_2}. \tag{4.2}$$

Proof. The set of characters χ with trivial stabiliser under the W-action is dense in $\operatorname{Spec}\mathbb{C}[\Lambda]$. Hence it suffices to prove the proposition for such χ. As in [Mc2, Theorem 7.1], we may use [BZ, Theorem 5.2] to conclude that $\dim \operatorname{Hom}_{\widetilde{G}}(I(\chi), I(w\chi)) = 1$. The proposition now follows from Theorem 4.2. $\square$

5. Iwahori invariants

The structure and results of this section closely follow [CO, §3.3]. Let I be the Iwahori subgroup of G, maximal with respect to intersection with P among all Iwahori subgroups contained in K.

Proposition 5.1. *The dimension of the space of vectors in an unramified principal series representation $I(\chi)$ invariant under the Iwahori subgroup I is equal to $|W|$.*

Proof. One has $\widetilde{P}\backslash\widetilde{G}/I \simeq W$. Thus $\dim(I(\chi)^I) \leq |W|$.

For any $w \in W$, we can define $\phi_w \in I(\chi)^I$ by $\phi_w(g) = \phi_K(g)$ if $g \in \widetilde{P}wI$ and $\phi_w(g) = 0$ otherwise. These functions are obviously linearly independent so the proposition is proved. $\square$

Let w_0 denote the longest element in W. Let $\phi_{w_0}^{(\chi)}$ be the function in $I(\chi)$ supported on $\widetilde{P}w_0I$ and taking the value v_0 at w_0.

Proposition 5.2. [CO, Lemma 5] *A basis for the space $I(\chi)^I$ can be given by the elements $T_w\phi_{w_0}^{(w^{-1}\chi)}$ as w ranges over W.*

Proof. Let us write f_w for the function $T_w\phi_{w_0}^{(w^{-1}\chi)}$. Suppose that $f_w(v)\neq 0$ for some $v\in W$. Recall we have

$$f_w(v)=\int_{U_w}\phi_{w_0}^{(w^{-1}\chi)}(w^{-1}uv)du.$$

For this to be nonzero, it is necessary that $w^{-1}Uv\cap \widetilde{P}w_0I\neq\emptyset$. Since the group I admits an Iwahori factorisation, we have the containment $\widetilde{P}w_0I\subset\widetilde{P}w_0\widetilde{P}$. Now $w^{-1}Uv\subset Pw^{-1}PvP$ and since these are double cosets in a Tits system, this intersects Pw_0P non-trivially only if $\ell(w^{-1})+\ell(v)\geq\ell(w_0)$. Furthermore, if there is equality $\ell(w^{-1})+\ell(v)=\ell(w_0)$, the intersection is non-empty if and only if $w^{-1}v=w_0$.

From the above considerations, it suffices to show that $f_w(ww_0)\neq 0$. Again unravelling the definitions, we need to know when $w^{-1}uww_0\in Pw_0I$ with $u\in U$. We rewrite this as $w^{-1}uw\in P\cdot w_0(I\cap U)w_0^{-1}$. The part of $w^{-1}uw$ lying in P can be ignored, since it corresponds to u lying in wUw^{-1}, which is quotiented out in the definiton of U_w. Thus we are essentially integrating over u lying in a compact set of positive measure. For such u, $\phi_{w_0}(w^{-1}uww_0)=v_0$, so $f_w(ww_0)\neq 0$, as required. □

Proposition 5.3. *The spherical function ϕ_K can be expanded as*

$$\phi_K=\sum_{w\in W}c_{w_0}(w^{-1}\chi)\overline{T}_w\phi_{w_0}^{(w^{-1}\chi)}.$$

Proof. By the previous proposition, there exist $d_w(\chi)$ such that

$$\phi_K^{(\chi)}=\sum_{w\in W}d_w(\chi)\overline{T}_w\phi_{w_0}^{(w^{-1}\chi)}. \tag{5.1}$$

Let us apply $\overline{T}_u$ to this equation. Via (4.1) and (4.2), we arrive at

$$\sum_{w\in W}d_w(u\chi)\overline{T}_w\phi_{w_0}^{(w^{-1}u\chi)}=\sum_{w\in W}d_w(\chi)\overline{T}_{uw}\phi_{w_0}^{(w^{-1}u\chi)}.$$

Again using the fact we have a basis of $I(u\chi)^I$, we compare coefficients to obtain $d_w(\chi)=d_{uw}(u\chi)$. Thus it suffices to establish the value of $d_{w_0}(\chi)$. We now evaluate (5.1) at the identity.

$$\phi_K^{(\chi)}(1)=\sum_{w\in W}d_w(\chi)c_{w_0}(w\chi)^{-1}\int_{U_w}\phi_{w_0}^{(w^{-1}\chi)}(w^{-1}u)du.$$

As in the proof of Proposition 5.2, $w^{-1}u\in Pw_0I$ if and only if $w=w_0$ and $u\in U\cap I$. Thus only one term survives in this sum, and the surviving integral is the integral of v_0 over a set of measure one. We end up with $d_{w_0}(\chi)=c_{w_0}(w^{-1}\chi)$ which implies the result.

□

6. Whittaker functionals

We fix a character ψ of U that is unramified. By this, we mean that ψ is a homomorphism from U to $\mathbb{C}^\times$ with the following property. For each simple reflection s, the restriction of U to the intersection $U_s=U\cap M_s$ with the corresponding Levi subgroup M_s is trivial on $U_s\cap K$ and non-trivial on any open subgroup of U_s with a larger abelianisation than $U_s\cap K$.

Definition 6.1. *A Whittaker functional on a representation (π, V) of $\widetilde{G}$ is defined to be a linear functional W on V such that $W(\pi(u)v) = \psi(u)v$ for all $u \in U$ and $v \in V$.*

Theorem 6.2. *There is an isomorphism between $i(\chi)^*$ and the space of Whittaker functionals on $I(\chi)$ given by $\lambda \mapsto W_\lambda$ with*

$$W_\lambda \phi = \lambda \left(\int_{U^-} \phi(uw_0)\psi(u)du \right).$$

Proof. This follows from [BZ, Theorem 5.2]. □

Let us choose a set of coset representatives for the coset space $\widetilde{M}/H$. In the unramified case, without loss of generality, we may assume that all coset representatives are of the form ϖ^λ for some $\lambda \in X_*(S)$. This is not necessary, but will facilitate the computation at times. As a runs through such a set of coset representatives, the vectors $\pi_\chi(a)v_0$ form a basis of $i(\chi)$. We will write $\lambda_a^{(\chi)}$ for the corresponding dual basis of $i(\chi)^*$ and let $W_a^{(\chi)}$ be the Whittaker functional corresponding to $\lambda_a^{(\chi)}$ under the bijection of Theorem 6.2. The functional $\lambda_a^{(\chi)}$ depends only on a and not on the choice of a set of coset representatives including a.

We now introduce the change of basis coefficients as in [KP] that are fundamental to the major thrust of this paper.

For any a and any w, the composite $W_a^{(w\chi)} \circ \overline{T}_w$ is also a Whittaker functional on $I(\chi)$. Thus it can be expanded in any basis we have, so we define coefficients $\tau_{a,b}^{(w,\chi)}$ by

$$W_a^{(w\chi)} \circ \overline{T}_w = \sum_b \tau_{a,b}^{(w,\chi)} W_b^{(\chi)}.$$

We need to know how these coefficients change under a change of coset representatives. For $h \in H$, we have

$$\tau_{a,bh}^{(w,\chi)} = \chi(h)\tau_{a,b}^{(w,\chi)} \quad \text{and} \quad \tau_{ah,b}^{(w,\chi)} = \frac{\tau_{a,b}^{(w,\chi)}}{(w\chi)(h)}.$$

As a consequence, after choosing an extension of χ to a function on $\widetilde{M}$ satisfying $\chi(mh) = \chi(m)\chi(h)$ for $h \in H$, the quantity

$$\widetilde{\tau}_{a,b}^{(w,\chi)} = \frac{(w\chi)(a)}{\chi(b)} \tau_{a,b}^{(w,\chi)}$$

is independent of the choice of coset representatives, only depending on the cosets of a and b (and also depending on the choice of extension of χ).

We conclude this section with a useful lemma. We use $R(g)$ to denote the action of g on a function by right-translation, and say that $a \sim b$ if a and b are in the same H-coset. We write t^* for the conjugate of t under w_0.

Lemma 6.3. [CO, Lemma 7] *Let $t \in \widetilde{M}$. The expression $W_b^{(\chi)}(R(t)\phi_{w_0}^{(\chi)})$ vanishes unless t is dominant and $t^* \sim b$. In this latter case, it is equal to $\delta^{1/2}(t)\chi(t^* b^{-1})$.*

Proof. Unravelling the definitions, we have

$$W_b^{(\chi)}(R(t)\phi_{w_0}^{(\chi)}) = \lambda_b^{(\chi)} \int_{U^-} \phi_{w_0}^{(\chi)}(uw_0t)\psi(u)du.$$

Since the group I admits an Iwasawa decomposition, for any $u' \in U^-$, we have $u'w_0 \in \widetilde{P} w_0 I$ if and only if $u' \in U^-(O_F)$. We apply this to $u' = (t^*)^{-1} u t^*$, to obtain

$$W_b^{(\chi)}(R(t)\phi_{w_0}^{(\chi)}) = \lambda_b^{(\chi)} \delta^{1/2}(t^*) \pi_\chi(t^*) v_0 \int_{t^* U^-(O_F)(t^*)^{-1}} \psi(u) du.$$

The first factor vanishes unless $t^* \sim b$, when it is equal to $\delta^{1/2}(t^*)\chi(t^* b^{-1})$. The integral vanishes whenever ψ is a non-trivial character on the group $t^* U^-(O_F)(t^*)^{-1}$. This occurs unless t is dominant, in which case we get the relevant volume, namely $\delta(t)$, appearing as a factor. □

7. Constructing the Chinta-Gunnels action

Consider the matrices $D_w^{(\chi)} = (\tau_{a,b}^{(w,\chi)})_{a,b}$ and $\tilde{D}_w^{(\chi)} = (\tilde{\tau}_{a,b}^{(w,\chi)})_{a,b}$ formed from the coefficients introduced in the previous section. By Proposition 4.4, we easily calculate the cocycle condition

$$D_{w_1 w_2}^{(\chi)} = D_{w_1}^{(w_2\chi)} D_{w_2}^{(\chi)}. \tag{7.1}$$

Since $\widetilde{D}_w^{(\chi)} = T_{w\chi} D_w^{(\chi)} T_\chi^{-1}$ where T_χ is the diagonal matrix with entries $\chi(b)$, it satisfies the same cocycle condition

$$\widetilde{D}_{w_1 w_2}^{(\chi)} = \widetilde{D}_{w_1}^{(w_2\chi)} \widetilde{D}_{w_2}^{(\chi)}.$$

We know a priori from the definition that $\tau_{a,b}^{(w,\chi)}$ is a function on X, but do not yet know that it lies in B. It will turn out that this is indeed the case, following from similar considerations as in the proof of Theorem 4.2, once we have equation (9.2). Let us provisionally assume that it is the case that $\tau_{a,b}^{(w,\chi)} \in B$. This will not cause any circularity in the arguments presented.

Write $\Gamma = \widetilde{M}/H$ and let A be the fraction field of $\mathbb{C}[\Lambda']$. Consider the standard componentwise action of W on A^Γ. Using the cocycle $\widetilde{D}$ we may twist this action to define a new action of W on A^Γ, we denote it by $(w, f) \mapsto w \circ f$. Explicitly

$$(w \circ f)(w\chi) = \widetilde{D}_w^{(\chi)} f(\chi)$$

for all $f \in A^\Gamma$ and $w \in W$. We are thinking of the field A here as the field of rational functions on $\operatorname{Spec}(\mathbb{C}[\Lambda'])$.

There is a canonical isomorphism of B-modules $A \simeq B^\Gamma$. We will write π_γ for the projection onto the factor with index γ. By taking the tensor product with A over B and using the natural inclusion of B in A, this induces a canonical injection of A-modules $j : A \hookrightarrow A^\Gamma$. Explicitly $j(g) = (\pi_\gamma(g))_{\gamma\in\Gamma}$.

Proposition 7.1. *The image of j is invariant under the action of W.*

Proof. We write

$$(w \circ j(g))_a(w\chi) = \sum_b \frac{(w\chi)(a)}{\chi(b)} \tau_{a,b}^{(w,\chi)}(\pi_b(g))(\chi).$$

Now $\pi_b(g) \in B_b$. Multiplication by the factor $\frac{(w\chi)(a)}{\chi(b)}$ lands us in $B_{w^{-1}a}$. Since $\tau_{a,b}^{(w,\chi)} \in B$, the right hand side of the above equation is an element of $B_{w^{-1}a}$ and hence $(w \circ j(g))_a \in B_a$ proving that $w \circ j(g)$ lies in the image of j. □

As a consequence of this proposition, restricting our action of W on A^Γ to the image of j defines an action of W on A. We may be rather explicit and write

$$(w \circ g)(w\chi) = \sum_{a,b} \widetilde{\tau}_{a,b}^{(w,\chi)} \pi_b(g)(\chi). \tag{7.2}$$

It will eventually transpire that for split groups G, this action will agree with the action constructed in [CG], hence the title of this section. It is possible to be more precise and find smaller subrings of A stabilised by this action, but this shall not concern us.

8. Formal computation of the Whittaker function

There is another basis of $i(\chi)^*$ in the unramified case that is more amenable to calculation than the one we have so far considered. It is parametrised by extensions of χ to a character $\tilde{\chi}$ of $X_*(S)$, and only depends on the choice of a lift of the lattice $X_*(S)$ to a subgroup of $\widetilde{G}$. In this way, given a coset representative as in the previous section of the form ϖ^λ, we can meaningfully talk about $\tilde{\chi}(a)$.

In general, we have to be more circumspect, and do not get anything approaching a distinguished choice of basis. As in the previous section, we consider an extension $\tilde{\chi}$ of χ to a function on $\widetilde{M}$ satisfying $\tilde{\chi}(mh) = \tilde{\chi}(m)\chi(h)$ for $m \in \widetilde{M}$ and $h \in H$. We consider the functional $\lambda_{\tilde{\chi}}$ on $i(\chi)$ defined by $\lambda_{\tilde{\chi}}(\pi_\chi(a)v_0) = \tilde{\chi}(a))$, and will compute the corresponding Whittaker function.

In the unramified case, choosing $\tilde{\chi}(\varpi^\lambda) = \tilde{\chi}(\lambda)$ yields a basis of $i(\chi)^*$ as $\tilde{\chi}$ runs over the set of all extensions of χ to $X_*(S)$. Let us now fix for once and all a particular extension $\tilde{\chi}$ of χ. By abuse of notation, we will simply write χ for this extension throughout.

The Whittaker function which we we are aiming to compute is the function

$$\mathcal{W}_\chi(g) = W_{\lambda_\chi}^{(\chi)}(R(g)\phi_K).$$

It is a complex-valued function on $\widetilde{G}$ satisfying

$$\mathcal{W}_\chi(\zeta u g k) = \zeta\psi(u)\mathcal{W}_\chi(g)$$

for all $\zeta \in \mu_n$, $u \in U$, $g \in \widetilde{G}$ and $k \in K$. The Iwasawa decomposition takes the form $G = UMK$, so in order to compute $\mathcal{W}_\chi$, it suffices to know the values taken by $\mathcal{W}_\chi$ on $\widetilde{M}$. This is what we shall concentrate our efforts on.

For $t \in \widetilde{M}$, write m_t for the function $m_t(\chi) = \chi(w_0 t w_0^{-1})$.

Theorem 8.1. *Suppose $t \in \widetilde{M}$. The Whittaker function $\mathcal{W}_\chi(t)$ vanishes unless t is dominant. If t is dominant, then we have*

$$\mathcal{W}_\chi(t) = \delta^{1/2}(t) \sum_{w \in W} c_{w_0}(w^{-1}\chi)(w \circ m_t)(\chi).$$

Proof. The proof is a combination of the various results we have to date. We begin by unravelling our formula for this Whittaker function using Lemma 5.3.

$$
\begin{aligned}
\mathcal{W}_\chi(t) &= \sum_a \chi(a) W_a^{(\chi)}(R(t)\phi_K) \\
&= \sum_a \chi(a) W_a^{(\chi)}(R(t)\sum_w c_{w_0}(w^{-1}\chi)\overline{T}_w \phi_{w_0}^{(w^{-1}\chi)}) \\
&= \sum_{a,w} \chi(a) c_{w_0}(w^{-1}\chi) W_a^{(\chi)} \overline{T}_w R(t)\phi_{w_0}^{(w^{-1}\chi)} \\
&= \sum_{a,b,w} \chi(a) c_{w_0}(w^{-1}\chi) \tau_{a,b}^{(w,w^{-1}\chi)} W_b^{(w^{-1}\chi)} R(t)\phi_{w_0}^{(w^{-1}\chi)}.
\end{aligned}
$$

To continue, we apply Lemma 6.3. Unless t is dominant, this tells us that our Whittaker function vanishes. When t is dominant, we continue our manipulation which will deposit us at our desired result.

$$
\begin{aligned}
\mathcal{W}_\chi(t) &= \sum_{a,b,w} \chi(a) c_{w_0}(w^{-1}\chi) \tau_{a,b}^{(w,w^{-1}\chi)} \delta_{b\sim t^*} \delta^{1/2}(t)(w^{-1}\chi)(t^* b^{-1}) \\
&= \delta^{1/2}(t) \sum_{a,b,w} c_{w_0}(w^{-1}\chi) \widetilde{\tau}_{a,b}^{(w,w^{-1}\chi)} \pi_b(m_t)(w^{-1}\chi) \\
&= \delta^{1/2}(t) \sum_{w\in W} c_{w_0}(w^{-1}\chi)(w\circ m_t)(\chi).
\end{aligned}
$$

□

9. Setup for explicit computation

In this section we lay the groundwork for the explicit calculation for unramified G that will follow. As a side effect, we will also be able to provide a proof of the missing claim that $\tau_{a,b}^{(w,\chi)} \in B$.

We begin by choosing an open compact subgroup K_1 of G normalised by W and admitting an Iwahori factorisation with respect to P. For each $b \in \widetilde{S}$, define $f_b \in I(\chi)$ to be the unique function in this space supported on $\widetilde{P} w_0 K_1$ and taking the value $\pi_\chi(b)v_0$ at w_0.

Lemma 9.1. *We have $W_a^{(\chi)}(f_b) = 0$ unless $a \sim b$, in which case $W_a^{(\chi)}(f_a) = |U^- \cap K_1|$.*

Proof. Consider the defining integral for $W_a^{(\chi)}$. Since K_1 is assumed to admit an Iwahori factorisation, $PK_1 \cap U^- = K_1 \cap U^-$. The remainder of the verification is routine. □

Corollary 9.2.

$$
\tau_{a,b}^{(w,\chi)} = \frac{(W_a^{(w\chi)} \circ \overline{T}_w)(f_b)}{|U^- \cap K_1|}.
$$

Our aim is to compute explicitly the action $w \circ -$ for a simple reflection in W. Thus we take $w = s = s_\alpha$, a simple reflection corresponding to the simple coroot α of G'. Expanding the integrals in the above Corollary yield

$$
\tau_{a,b}^{(s,\chi)} = \frac{1}{c_s(\chi)|U^- \cap K_1|} \lambda_a^{(s\chi)} \left(\int_{U^-} \int_{U_s} f_b(s^{-1} n u w_0) dn \psi(u) du \right). \tag{9.1}
$$

Lemma 9.3. *Suppose that $n \in U_s$ is not equal to the identity, and $u \in U^-$. Then there is a unique $n' \in U_s^-$ such that $p(n) := s^{-1}nn' \in \widetilde{P}$. Furthermore, $s^{-1}nuw_0 \in \widetilde{P}w_0K_1$ if and only if $u = n'u'$ with $u' \in U^- \cap K_1$.*

Proof. The first claim is an immediate consequence of the Bruhat decomposition in the rank one group M_s. For the latter claim, write $u = n'u'$. Then $s^{-1}nuw_0 \in \widetilde{P}w_0K_1$ if and only if $p(n)u' \in \widetilde{P}K_1$ and this set is equal to $\widetilde{P}(U^- \cap K_1)$ since K_1 admits an Iwasawa decomposition. This finishes the proof. □

As an immediate consequence of this lemma, we simplify (9.1) to leave ourselves with an expression for the coefficient $\tau_{a,b}^{(s,\chi)}$ with only one integral, namely

$$\tau_{a,b}^{(s,\chi)} = c_s(\chi)^{-1} \int_{U_s} \lambda_a^{(s\chi)} f_b(p(n)w_0)\psi^{-1}(n')dn \tag{9.2}$$

Since the subset $\{1\} \subset U_s$ where the integrand is not defined is of measure zero, it may safely be ignored.

Using this equation, we are able to justify the claim that $\tau_{a,b}^{(w,\chi)} \in B$ that was promised. We argue in an analogous manner to the proof of Theorem 4.2 to cover the case where w is a simple reflection. The case of general w then follows from the cocycle relation (7.1).

10. Digression on SU_3

From now until the end of the paper we will assume that G is an unramified group. For any simple reflection s, we let G_s be the simply connected cover of the derived group of M_s. The group G_s is a simply-connected, semisimple unramified group of rank one, and such groups are completely classified. There are two possibilities, either G_s is isomorphic to $\mathrm{Res}_{E/F}SL_2$ for an unramified extension E of F, or is isomorphic to $\mathrm{Res}_{E/F}SU_3$, where the special unitary group SU_3 over E is defined in terms of an unramified quadratic extension L of E, which again is unramified over F.

Of these two possibilities, the group SL_2 will be familiar to most readers. We pause to collate some facts about the less well-known SU_3 that will prove to be of use later on.

We use $\overline{z}$ to denote the image of z under the non-trivial element of $\mathrm{Gal}(L/E)$. Pick $\theta \in L$ such that $|\theta| = 1$ and $\theta + \overline{\theta} = 0$. The special unitary group $SU_3(E)$ is defined to be the subgroup of $SL_3(L)$ preserving the Hermitian form $x_1\overline{x_3} + x_2\overline{x_2} + x_3\overline{x_1}$. Explicitly, if J is the matrix with ones on the off-diagonal and zeroes elsewhere, then $X \in SL_3(L)$ is in $SU_3(E)$ if and only if ${}^t\overline{X}JX = J$.

These coordinates are chosen such that the intersection of $SU_3(E)$ with the set of upper-triangular matrices constitutes a Borel subgroup. Its unipotent radical consists of all matrices of the form

$$u = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix}$$

where x and y are elements of L with $x\overline{x} + y + \overline{y} = 0$.

We may take the maximal compact subgroup K to be the subgroup consisting of all matrices with entries in O_L.

For any $r \in \mathbb{R}$, the set of $u \in U$ with $v(y) \geq r$ forms a subgroup of U. (This is the filtration induced by a valuation of root datum in Bruhat-Tits theory introduced in the proof of Theorem 4.2). Let us denote this subgroup by U_r.

A particular aspect that will require some care, is that the fibres of the map $u \mapsto \varpi^{-2m} y$ from $U_{2m} \setminus U_{2m-1}$ to $O_L^\times$ do not all have the same volume. Namely the volume of a fibre over a point z with $z + \overline{z} \in O_L^\times$ is $q+1$ times the volume of the fibre over a point z with $z + \overline{z} \in \varpi O_L$.

The following equation is fundamental, and explicitly realises the first part of Lemma 9.3 in $SU_3(E)$.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\overline{y} & x/y & 1 \\ 0 & \overline{y}/y & \overline{x} \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \overline{x}/\overline{y} & 1 & 0 \\ \overline{y}^{-1} & -x/y & 1 \end{pmatrix}^{-1} \tag{10.1}$$

Later we will see how to lift this to an equation in the metaplectic cover.

Let us say that an element of U is of type I if $v(y) = 2v(x)$ and is of type II otherwise.

Let α be the positive generator of $X_*(S)$. We can, and do, write $\alpha = \alpha_1 + \alpha_2$ where α_1 and α_2 are simple coroots of SL_3.

11. The descent process

Since we are only working with unramified groups, Galois descent for an unramified extension of local fields shall be the only descent process that shall concern us. We will now give a discription of the construction of the metaplectic group $\widetilde{G}$ in a manner that clarifies the relationship with such Galois descent. This overview is described more precisely in [BD, §12.11].

First one constructs the central extension of $G(F)$ by $K_2(F)$. Then one pushes forward by the tame symbol $K_2(F) \to k^\times$ to arrive an an extension of G by $k^\times$. One finally pushes forward this extension via the operation of raising to the $(q-1)/n$-th power to obtain the metaplectic group $\widetilde{G}$.

Now suppose that E is a degree d unramified extension of F. This description of $\widetilde{G}$ is particularly amenable to descent and shows that $\widetilde{G}$ can be realised via descent as a subgroup of a group $\widetilde{G(E)}$ which is a central extension of $G(E)$ by the group of $n\frac{q^d-1}{q-1}$-th roots of unity. In fact this description has already been implicitly used in the discussion following Assumption 2.3 to justify the use of this assumption in the unramified case.

We will first consider how descent behaves for a restriction of scalars for semisimple groups. The answer we will get is as expected. Namely, if $G = \mathrm{Res}_{E/F} G'$, then the metaplectic group $\widetilde{G}$ is isomorphic to the central extension of $G(F) = G'(E)$ by μ_n obtained by considering G' as an algebraic group over E.

To check this, it suffices to restrict to a maximal torus of G. Let us write Γ for the Galois group $\mathrm{Gal}(E/F)$. Let T' be a maximal torus of G' and $T = \mathrm{Res}_{E/F}(T')$, which is a maximal torus of G. As Galois modules we have $X_*(T) = X_*(T') \otimes \mathbb{Z}[\Gamma]$.

Consider $T'(E) = T(F) \hookrightarrow T(E)$. At the level of cocharacter lattices this corresponds to the diagonal embedding $X_*(T') \hookrightarrow X_*(T') \otimes \mathbb{Z}[\Gamma]$, namely $y \mapsto \sum_{\gamma \in \Gamma} y \otimes \gamma$. Let us write Q' for the restriction of Q to $X_*(T') \otimes 1$.

If $\gamma_1 \neq \gamma_2$, then Weyl group invariance of Q implies that

$$B(y_1 \otimes \gamma_1, y_2 \otimes \gamma_2) = \frac{1}{|W|} B(y_1 \otimes \gamma_1, \sum_{w \in W} w y_2 \otimes \gamma_2) = 0$$

where to make the last identification we used the fact that G is semisimple.

As a consequence, the quadratic form Q is completely determined by Q', due to its Weyl and Galois-invariance. So if $t_1, t_2 \in T'(E)$, we may calculate using an explicit incarnation of our cocycle that

$$\sigma_T^Q(t_1, t_2) = \prod_{\gamma \in \Gamma} \sigma^{Q'}(\gamma t_1, \gamma t_2) = \sigma_{T'}^{Q'}(t_1, t_2)^{\frac{q^d - 1}{q-1}}$$

which is enough to deduce that the desired behaviour occurs.

The other descent calculation we need to study in detail is descent from SL_3 to SU_3, since we will need to have the ability to explicitly calculate in the metaplectic cover of SU_3. Our strategy is to realise $\widetilde{SU_3(E)}$ as a subgroup of the $n(q+1)$-fold cover of $SL_3(L)$, with the same quadratic form characterising the extension in each case. There is an explicit cocycle for the cover of $SL_3(L)$ given to us by Banks, Levi and Sepanski. Their result [BLS, Theorem 7], together with the equations appearing in its proof provide an algorithmic method to multiply in $\widetilde{SL_3(L)}$. It provides us with a section $\mathbf{s}$ and a 2-cocycle σ for which multiplication is given by $\mathbf{s}(g_1 g_2) = s(g_1)s(g_2)\sigma(g_1, g_2)$. We caution the reader than upon restriction of $\mathbf{s}$ to $SU_3(E)$, the image does not lie in $\widetilde{SU_3(E)}$.

Our strategy for cicrumventing this problem to find explicit elements of $\widetilde{SU_3(E)}$ is to use Theorem 2.2 which states that all unipotent subgroups are canonically split in central extensions. Our aim is to use this fact to lift the identity (10.1) into an identity in the metaplectic cover.

By construction the section $\mathbf{s}$ canonically splits the group U of upper-triangular unipotent matrices in SL_3. Hence, the splitting of the lower-triangular unipotent subgroup U^- must be given by $u \mapsto \mathbf{s}(w_0)\mathbf{s}(w_0 u w_0)\mathbf{s}(w_0)$.

Let me write

$$n_1 = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} 1 & -x/y & \overline{y}^{-1} \\ 0 & 1 & \overline{x}/\overline{y} \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the results of Banks Levi and Sepanski referenced above, it may be computed that $\sigma(n_1 w_0 n_2, w_0) = (x, \overline{y}/y)$ and $\sigma(w_0, p(n)) = (y, \overline{y})$. Thus we have

$$\mathbf{s}(w_0)\mathbf{s}(n_1)\mathbf{s}(w_0)\mathbf{s}(n_2)\mathbf{s}(w_0) = (x, y/\overline{y})(y, \overline{y})\mathbf{s}(p).$$

There is a subtlety that needs to be taken care of. According to our construction in Section 2, the choice of representative for the simple reflection s is not the same as the element w_0 we have been using so far in this computation, but differs by a factor of θ^α. Now let us write $y = v\theta\varpi^m$. Then after one more (simpler) cocycle computation, we find that

$$s^{-1} n n' = (x, y/\overline{y})(y, \overline{y})(v, \varpi)^{-m} v^\alpha \varpi^{m\alpha} \pmod{U} \tag{11.1}$$

12. Gindikin-Karpelevic formula

Recall that for a simple reflection s, G_s is defined to be the simply connected cover of the derived group of the corresponding Levi subgroup M_s. Let q denote the cardinality of the residue field of E, where E is as in the classification of G_s. Let $x_\alpha = \chi(\varpi^\alpha)$ and $\epsilon = (-1)^{n_\alpha}$. We have the following refinement of Theorem 4.2.

Theorem 12.1. *Suppose that G is unramified, and the simple reflection s corresponds to the simple coroot α. Then*

$$c_s(\chi) = \begin{cases} \dfrac{1-q^{-1}x_a^{n_a}}{1-x_\alpha^{n_\alpha}} & \textit{if} \;\; G_s \cong SL_2(E) \\ \dfrac{(1+\epsilon q^{-1}x_\alpha^{n_\alpha})(1-\epsilon q^{-2}x_\alpha^{n_\alpha})}{1-x_\alpha^{2n_\alpha}} & \textit{if} \;\; G_s \cong SU_3(E) \end{cases}$$

Proof. The author has already written a proof in the SL_2 case [Mc1, Theorem 6.4], so we will not repeat the argument here. In the SU_3 case, we present this argument as a warm up for the more challenging computation of $\tau_{a,b}^{(s,\chi)}$ that will be subsequently performed in Section 14. We follow the strategy from the proof of Theorem 4.2. Hence we have to evaluate

$$\int_{U_s} \phi_K(s^{-1}u)du.$$

The integral over $U_s \cap K$ is trivially equal to v_0. For the rest of the integral, we make use of the calculations of the previous section which imply that if $u \in C_m$, then $\phi_K(s^{-1}u) = (v\overline{v}, \varpi)^{m/2}\phi_K(\varpi^{m\alpha})$.

First let us assume that n_α is odd. Then n_α divides $2k$ if and only if it divides k. The contribution from $v(y)$ even is thus

$$\sum_{l=1}^{\infty} q^{4ln_\alpha}(1-q^{-3})(q^{-2}x_\alpha)^{2ln_\alpha} = (1-q^{-3})\frac{x_\alpha^{2n_\alpha}}{1-x_\alpha^{2n_\alpha}}.$$

For $v(y)$ odd, write $v(y) = (2l+1)n_\alpha$. The contribution this time is

$$\sum_{l=0}^{\infty}(q-1)q^{(2l+1)n_\alpha-2}(q^{-2}x_\alpha)^{(2l+1)n_\alpha} = q^{-2}(q-1)\frac{x_\alpha^{n_\alpha}}{1-x_\alpha^{2n_\alpha}}.$$

Add 1 to these geometric series to get the desired result.

Now we turn to the case where n_α is even. For n_α even we can ignore the part where $v(y)$ is odd, which always gives zero contribution to the integral. So suppose $m = 2k$ is even. Here the integral over C_m is non-vanishing whenever n_α divides $2k$.

If $k = ln_\alpha$, then we get a contribution of

$$\sum_{l=1}^{\infty}(1-q^{-3})q^{4ln_\alpha}(q^{-2}x_\alpha)^{2ln_\alpha} = (1-q^{-3})\frac{x_\alpha^{2n_\alpha}}{1-x_\alpha^{2n_\alpha}}.$$

as before.

Now suppose n_α does not divide k, but does divide $2k$. Write $k = (l+1/2)n_\alpha$. We have to separate the domain of integration into Type I and Type II pieces to evaluate. Taking care of the subtleties in the volume calculation foreshadowed in Section 10, we get a contribution of

$$\frac{-q(q-1)}{q^3-1}(1-q^{-3})\sum_{l=0}^{\infty} q^{4k}(q^{-2}x_\alpha)^{2k} = \frac{-q^{-2}(q-1)x_\alpha^{n_\alpha}}{1-x_\alpha^{2n_\alpha}}$$

and the only remaining calculation is to add 1 to the two rational functions produced. □

13. THE SL_2 CASE

Suppose $G_s \cong SL_2(E)$, q is the cardinality of the residue field of E and α is the unique positive coroot. The following proposition is equivalent to [KP, Lemma I.3.3]. We give a different proof, which will serve as a template for the more involved SU_3 case in the following section. Given any integer t, we define the Gauss sum $\mathfrak{g}_{SL_2(E)}(t)$ to be

$$\int_{O_F^\times} (v,\varpi)^t(\psi(\frac{v}{\varpi}))dv$$

with a choice of Haar measure such that the total volume of $O_F^\times$ is $q-1$.

Proposition 13.1. *Suppose that $a=\varpi^\nu$ and $b=\varpi^\mu$. Then we can write $\tau_{a,b}^{(s,\chi)} = \tau_{a,b}^1 + \tau_{a,b}^2$ where*

$$\begin{aligned} \tau_{a,b}^1 &= 0 \quad \text{unless} \quad \nu \sim \mu \\ \tau_{a,b}^2 &= 0 \quad \text{unless} \quad s\nu \sim \mu - \alpha \end{aligned}$$

If $\nu = s\mu + \frac{B(\alpha,\mu)}{Q(\alpha)}\alpha$, then

$$\tau_{a,b}^1 = (1-q^{-1})\frac{x_\alpha^{n_\alpha\lceil \frac{B(\alpha,\mu)}{n_\alpha Q(\alpha)}\rceil}}{1-q^{-1}x_\alpha^{n_\alpha}}.$$

If $\nu = s\mu+\alpha$, then

$$\tau_{a,b}^2 = q^{-1}\mathfrak{g}_{SL_2(E)}(B(\alpha,\mu)-Q(\alpha))\frac{1-x_\alpha^{n_\alpha}}{1-q^{-1}x_\alpha^{n_\alpha}}.$$

Write $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $x = \varpi^{-m}v^{-1}$ where $v \in O_E^\times$ and $m \in \mathbb{Z}$. The analagous statement to (11.1) is $p(n) = (v,\varpi)^{mQ(\alpha)}v^\alpha\varpi^{m\alpha}u'$ for some $u' \in U$. We now calculate

$$\begin{aligned} f_b(p(n)w_0) &= (v,\varpi)^{mQ(\alpha)}\delta^{1/2}(\varpi^{m\alpha})\pi_\chi(v^\alpha\varpi^{m\alpha})f_b(w_0) \\ &= q^{-m}(v,\varpi)^{mQ(\alpha)}\pi_\chi(bv^\alpha\varpi^{ma})v_0 \\ &= q^{-m}(v,\varpi)^{mQ(\alpha)}[b,v^\alpha]\pi_\chi(v^\alpha b\varpi^{m\alpha})v_0 \\ &= q^{-m}(v,\varpi)^{mQ(\alpha)+B(\alpha,\mu)}\pi_\chi(\varpi^{\mu+m\alpha})v_0 \end{aligned}$$

where in the last line we have used the commutator relation and the fact that $\phi_K(v^\alpha) = \phi_K(1)$ as ϕ_K is spherical.

Recall that we are trying to evaluate

$$\tau_{a,b}^{(s,\chi)} = c_s(\chi)^{-1}\int_{U_s}\lambda_a^{(s\chi)}f_b(p(n)w_0)\psi^{-1}(n')dn$$

We decompose U_s into shells where $|x|$ is constant on each shell, and the above calculations show that

$$\tau_{a,b}^{(s,\chi)} = c_s(\chi)^{-1}q^{-1}\lambda_a^{(s\chi)}\sum_{m\in\mathbb{Z}}\pi_\chi(\varpi^{\mu+m\alpha})\phi_K(1)\int_{O_F^\times}(v,\varpi)^{mQ(\alpha)+B(\alpha,\mu)}\psi(-\varpi^m v)dv.$$

where a normalisation of Haar measure on $O_F^\times$ is chosen such that the group has volume $q-1$.

If $m \le -2$, then the integral over $O_F^\times$ vanishes. Let us now consider the case $m = -1$. Here, the presence of the term $\lambda_a^{(s\chi)}\pi_\chi(\varpi^{\mu+m\alpha})v_0$ implies that this contribution is non-zero only when $s\nu \sim \mu - \alpha$. For $s\mu = \nu - \alpha$ we get a contribution of $c_s(\chi)^{-1}q^{-1}\mathfrak{g}_{SL_2(E)}(B(\alpha,\mu) - Q(\alpha))$ by the definition of the Gauss sum. This takes care of the $\tau^2_{a,b}$ part of the proposition.

Now we turn to the case where $m \ge 0$. In such a case, the argument of ψ is in O_E, so the character ψ automatically takes the value 1. Hence the integral vanishes unless it is identically equal to 1, which occurs if and only if $B(\alpha,\mu) + mQ(\alpha) \equiv 0 \pmod{n}$.

It is a standard fact about root systems that $B(\alpha,\mu)$ is divisible by $Q(\alpha)$. Thus the condition for non-vanishing of the integral now becomes $m \equiv -B(\alpha,\mu)/Q(\alpha) \pmod{n_\alpha}$. Let us write $m = kn_\alpha - B(\alpha,\mu)/Q(\alpha)$. Then k runs over all integers greater than or equal to $\lceil \frac{B(\alpha,\mu)}{n_\alpha Q(\alpha)} \rceil$. For m in this family, all elements of the form $\varpi^{\mu+m\alpha}$ lie in the same H-coset. Hence we get a contribution of zero unless $s\nu \sim \mu - \frac{B(\alpha,\mu)}{Q(\alpha)}\alpha$. From the definition of the action of W on the cocharacter lattice, this is equivalent to $\nu \sim \mu$.

If indeed we do have $\nu \sim \mu$, then the sum turns into a geometric series, namely it is equal to $c_s(\chi)^{-1}(1-q^{-1})\sum_k x_\alpha^{kn_\alpha}$. Theorem 12.1 then completes the proof.

14. The SU_3 case

Now that we have warmed up by proving a Gindikin-Karpelevic formula and covered the computation of the coefficients $\tau_{a,b}^{(s,\chi)}$ in the SL_2 case, we present the (completion of) the major component of this work, the version of Proposition 13.1 for the SU_3 case. Specifically, we assume now that $G_s \cong \mathrm{Res}_{E/F} SU_3$, let q be the cardinality of the residue field of E and let α be the unique positive rational coroot. Given any integer t, we define the Gauss sum $\mathfrak{g}_{SU_3(E)}(t)$ to be the sum

$$\mathfrak{g}_{SU_3(E)}(t) = \sum_{u \in U(\mathbb{F}_q)\backslash 1} (y\overline{y}, \varpi)^t \psi(\frac{x}{\varpi y})$$

where $u = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix} \in U(\mathbb{F}_q)$. The author does not know any properties of this particular algebraic integer analogous to the SL_2 case.

Proposition 14.1. *Suppose that $a = \varpi^\nu$ and $b = \varpi^\mu$. Then we can write $\tau_{a,b}^{(s,\chi)} = \tau^1_{a,b} + \tau^2_{a,b}$ where*

$$\begin{aligned} \tau^1_{a,b} &= 0 \quad \text{unless} \quad \nu \sim \mu \\ \tau^2_{a,b} &= 0 \quad \text{unless} \quad s\nu \sim \mu - 2\alpha \end{aligned}$$

If $\nu = s\mu + \frac{B(\alpha,\mu)}{Q(\alpha)}\alpha$, then

$$\tau^1_{a,b} = \frac{(1-q^{-3})x_\alpha^{2n_\alpha \lceil \frac{B(\alpha,\mu)}{2n_\alpha Q(\alpha)} \rceil} + (q^{-1}-q^{-2})q^{(1-\epsilon)/2} x_\alpha^{(2\lceil \frac{B(\alpha,\mu)+n_\alpha Q(\alpha)-Q(\alpha)}{2n_\alpha Q(\alpha)} \rceil - 1)n_\alpha}}{(1-\epsilon q^{-1}x_\alpha^{n_\alpha})(1+\epsilon q^{-2}x_\alpha^{n_\alpha})}$$

If $\nu = s\mu + 2\alpha$*, then*

$$\tau^2_{a,b} = q^{-2}\mathfrak{g}_{SU_3(E)}(B(\alpha,\mu)/2 - Q(\alpha))\frac{1 - x_\alpha^{2n_\alpha}}{(1-\epsilon q^{-1}x_\alpha^{n_\alpha})(1+\epsilon q^{-2}x_\alpha^{n_\alpha})}.$$

Proof. We continue to use all notations from previous sections without further explanation. In particular, let us recall that $p(n) = (v,\overline{v})^{mQ(\alpha)/2}v^{\alpha_1}\overline{v}^{\alpha_2}\varpi^{m\alpha}$ times an element of U. The commutator of $v^{\alpha_1}\overline{v}^{\alpha_2}$ and ϖ^μ is $(v\overline{v},\varpi)^{B(\alpha,\mu)/2}$. Thus the analagous evaluation of the function f_b is

$$f_b(p(n)w_0) = q^{-2m}(v\overline{v},\varpi)^{(mQ(\alpha)+B(\alpha,\mu))/2}\pi_\chi(\varpi^{\mu+m\alpha})v_0.$$

Let $x = yz$. Then $1/y = -z\overline{z}/2 + h\theta$ for some $h \in E$.

Let us first suppose that we are in case II and $v(y) \geq 3$. Then since we're in Case II, the Hilbert symbol $(v\overline{v},\varpi)$ only depends on h (and not on z). Thus from our integral (9.2), we may factor out an integral of $\psi(z)$ over the coordinate z. Since $v(y) \geq 3$, we have that z is running over an additive subgroup of L that is at least $\varpi^{-1}O_L$ so we get a contribution of zero.

Now suppose that we're in Case I and $v(y) \geq 3$. In this case z is running over a subgroup that is at least as big as $\varpi^{-2}O_L$. Let us pick any $u \in O_E^\times$ and consider the measure-preserving homeomorphism $z \mapsto z + \varpi^{-1}u$. This leaves the domain of integration invariant, and multiplies the integrand by $\psi(u/\varpi)$ which in general is not equal to 1. Hence such y also contribute zero to the τ coefficient.

Now let us turn to the case where $v(y) = 2$. For this we have a contribution of

$$c_s(\chi)^{-1}q^{-2}\lambda_a^{(s\chi)}\pi_\chi(\varpi^{\mu-2\alpha})v_0\int_{C_2}(v\overline{v},\varpi)^{B(\alpha,\mu)/2-Q(\alpha)}\psi(x/y)du.$$

This clearly vanishes unless $s\nu \sim \mu - 2\alpha$ and when $\nu = s\mu + 2\alpha$, we get the desired

$$c_s(\chi)^{-1}q^{-2}\mathfrak{g}_{SU_3(E)}(B(\alpha,\mu)/2 - Q(\alpha)).$$

This completes the $\tau^2_{a,b}$ part of the Proposition.

Now let us turn our attention to the $v(y) \leq 1$ part of the domain of integration. In this reigime, the argument of ψ is guaranteed to lie in O_L, so our computation essentially reduces to that already carried out in the proof of the Gindikin-Karpelevic formula.

Let us note that $Q(\alpha)$ divides $B(\alpha,\mu)/2$. This is because $\alpha = \alpha_1 + \alpha_2$ where α_1 and α_2 are in the same Galois orbit. Thus by linearity of B and Galois-invariantness of μ, we have $B(\alpha,\mu) = 2B(\alpha_1,\mu)$ and we already know that $B(\alpha_1,\mu)$ is divisible by $Q(\alpha)$.

First, let us assume that n_α is odd.

If $m = 2k$ is even then the integral is non-zero if and only if it is identically one. This occurs if and only if $k \equiv -B(\alpha,\mu)/(2Q(\alpha)) \pmod{n_\alpha}$. So we get a zero contribution unless $s\nu \sim \mu - B(\alpha,\mu)/Q(\alpha)\alpha$. Again, this is equivalent to $\nu \sim \mu$

If we have $\nu = \mu$, then $k = ln_\alpha - B(\alpha,\mu)/(2Q(\alpha))$ and we get a geometric series, specifically

$$c_s(\chi)^{-1}(1-q^{-3})\sum_l x_\alpha^{2n_\alpha} = (1-q^{-3})\frac{x_\alpha^{2n_\alpha\lceil\frac{B(\alpha,\mu)}{2n_\alpha Q(\alpha)}\rceil}}{(1+q^{-1}x_\alpha^{n_\alpha})(1-q^{-2}x_\alpha^{n_\alpha})}.$$

For the case with m odd, the condition for non-vanishing is $n|mQ(\alpha) + B(\alpha,\mu)$. Since m is odd, we have $m = 2ln_\alpha - n_\alpha - B(\alpha,\mu)/Q(\alpha)$ is greater than or equal

to -1. So we get non-vanishing only when $s\nu \sim \mu - B(\alpha,\mu)/Q(\alpha)\alpha$, that is when $\nu \sim \mu$. When $\nu = \mu$, then our geometric series becomes

$$c_s(\chi)^{-1}(q^{-1}-q^{-2})\sum_l x_\alpha^{(2l-1)n_\alpha} = (q^{-1}-q^{-2})\frac{x_\alpha^{(2\lceil \frac{B(\alpha,\mu+n_\alpha Q(\alpha)-Q(\alpha)}{2n_\alpha Q(\alpha)}\rceil -1)n_\alpha}}{(1+q^{-1}x_\alpha^{n_\alpha})(1-q^{-2}x_\alpha^{n_\alpha})}.$$

This completes the proof for n_α odd.

Now suppose that n_α is even. Thus $m=2k$ is even and the exponent of $(v\overline{v},\varpi)$ in our integrand is $kQ(\alpha)+B(\alpha,\mu)/2$. Thus, a non-vanishing contribution occurs for only for $s\nu \sim \mu - B(\alpha,\mu)/Q(\alpha)\alpha$. When this exponent is divisible by n, we proceed as in the case for n_α odd, obtaining a contribution of

$$(1-q^{-3})\frac{x_\alpha^{2n_\alpha\lceil\frac{B(\alpha,\mu)}{2n_\alpha Q(\alpha)}\rceil}}{(1-q^{-1}x_\alpha^{n_\alpha})(1+q^{-2}x_\alpha^{n_\alpha})}.$$

Now suppose that n_α does not divide $k+B(\alpha,\mu)/(2Q(\alpha))$ but does divide $2k+B(\alpha,\mu)/Q(\alpha)$. Now we proceed as in the proof of Theorem 12.1. We get a contribution of

$$(q^{-2}-q^{-3})\frac{x_\alpha^{(2\lceil \frac{B(\alpha,\mu+n_\alpha Q(\alpha)-Q(\alpha)}{2n_\alpha Q(\alpha)}\rceil -1)n_\alpha}}{(1-q^{-1}x_\alpha^{n_\alpha})(1+q^{-2}x_\alpha^{n_\alpha})}.$$

This completes the proof.

□

15. COMPARISON

We conclude this paper by comparing our above results with those of Chinta and Gunnells [CG] on the construction of the local part of a Weyl group multiple Dirichlet series. To do so, we suppose that G is split, simple and simply connected, and that Q is chosen to take the value 1 on short coroots.

By (7.2) and Proposition 13.1, we have the following formula for the action of the simple reflection s_α on the monomial function $m_\lambda(\chi)=\chi(\varpi^\lambda)$.

$$(s_\alpha \circ m_\lambda)(\chi) = \frac{m_\lambda(s_\alpha\chi)}{1-q^{-1}x_\alpha^{n_\alpha}}\Big((1-q^{-1})x_\alpha^{n_\alpha\lceil\frac{B(\alpha,\lambda)}{n_\alpha Q(\alpha)}\rceil-\frac{B(\alpha,\lambda)}{Q(\alpha)}} \\ + q^{-1}x_\alpha^{-1}(1-x_\alpha^{n_\alpha})\mathfrak{g}_{SL_2(F)}(B(\alpha,\lambda)-Q(\alpha))\Big)$$

Let us now compare this to the action defined by Chinta and Gunnells. As per [CO, §9], we will need to make a minor change of variables from the Chinta-Gunnels paper to eliminate extraneous powers of q. In [CG, Definition 3.1], an action is defined for any dominant λ. We write $f \mapsto f|_\lambda w$ for this action. Then $f||w := x^\lambda(x^{-\lambda}f|_\lambda w)$ is independent of λ and defines an action of W on A. This action does not quite agree on the nose with that constructed in Section 7, instead we have

Proposition 15.1. *The two Weyl group actions $f \mapsto \frac{c_{w_0}(w^{-1}\chi)}{c_{w_0}(\chi)}(w\circ f)$ and $f \mapsto$ sgn $(w)\prod_{\alpha\in\Phi(w)} x_\alpha^{n_\alpha} f||w$ are the same.*

Proof. Note that if $b\in B$, then $w\circ(bf)=(wb)(w\circ f)$ and similarly with $\circ$ replaced by $||$. With this observation we check that the two claimed actions are indeed actions. Now to prove the proposition, we merely have to consider the case of w a simple

reflection. This is now a simple calculation since we have explicit formulae for both sides when f is a monomial, and both actions extend by linearity in f. □

This result also suggests how to extend the results in [CG] to the non-split case. We note that some of the content of this paper would accomplish some of the necessary work to achieve this aim. For example, we have obtained an independent (albeit rather indirect) proof of [CG, Theorem 3.2].

Chinta and Gunnells use their action to construct the p-part of a Weyl group multiple Dirichlet series. This requires the construction of an auxiliary polynomial

$$N(\chi,\lambda)=\prod_{\alpha>0}\frac{1-q^{-1}x_\alpha^{n_\alpha}}{1-x_\alpha^{n_\alpha}}\sum_{w\in W}\operatorname{sgn}(w)\left(\prod_{\alpha\in\Phi(w)}x_\alpha^{n_\alpha}\right)(1|_\lambda w)(\chi).$$

Now the culmination of the above leads to our final result. Informally, this states that the value of the metaplectic Whittaker function on a torus element is equal to the p-part of a Weyl group multiple Dirichlet series.

Theorem 15.2. *Let λ be dominant. The following identity holds:*

$$(\delta^{-1/2}\mathcal{W}_\chi)(\varpi^\lambda)=\chi(\varpi^\lambda)N(\chi,\lambda).$$

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